



6.4.7 Row operations

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Introduction

- In this topic, we will
 - Review the operations we can perform on a system of linear equations
 - Find the equivalent operations on augmented matrices
 - These will be called row operations
 - See that row operations can be undone
 - Introduce the concept of row equivalence
 - See that row equivalent augmented matrices have the same solution





Row-echelon form

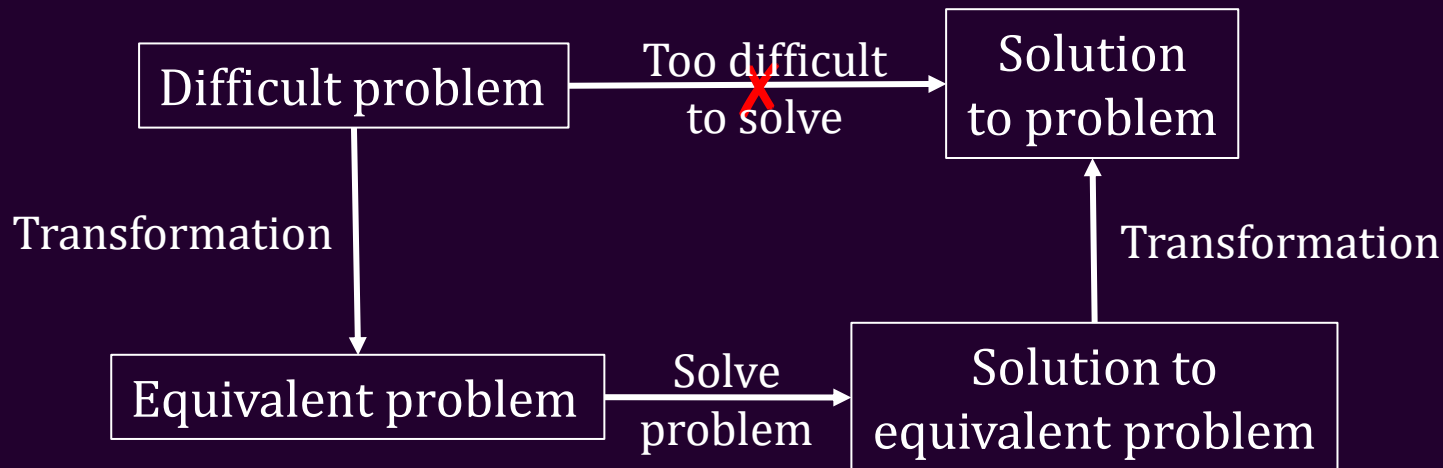
- Currently:
 - We can solve a system of linear equations if the corresponding augmented matrix is in row-echelon form
 - We know if there:
 - Are no solution
 - Is one unique solution
 - Are infinitely many solutions
 - We have seen algorithms for finding those solutions
- Question:
 - What if the augmented matrix is not in row-echelon form?





Transforming problems

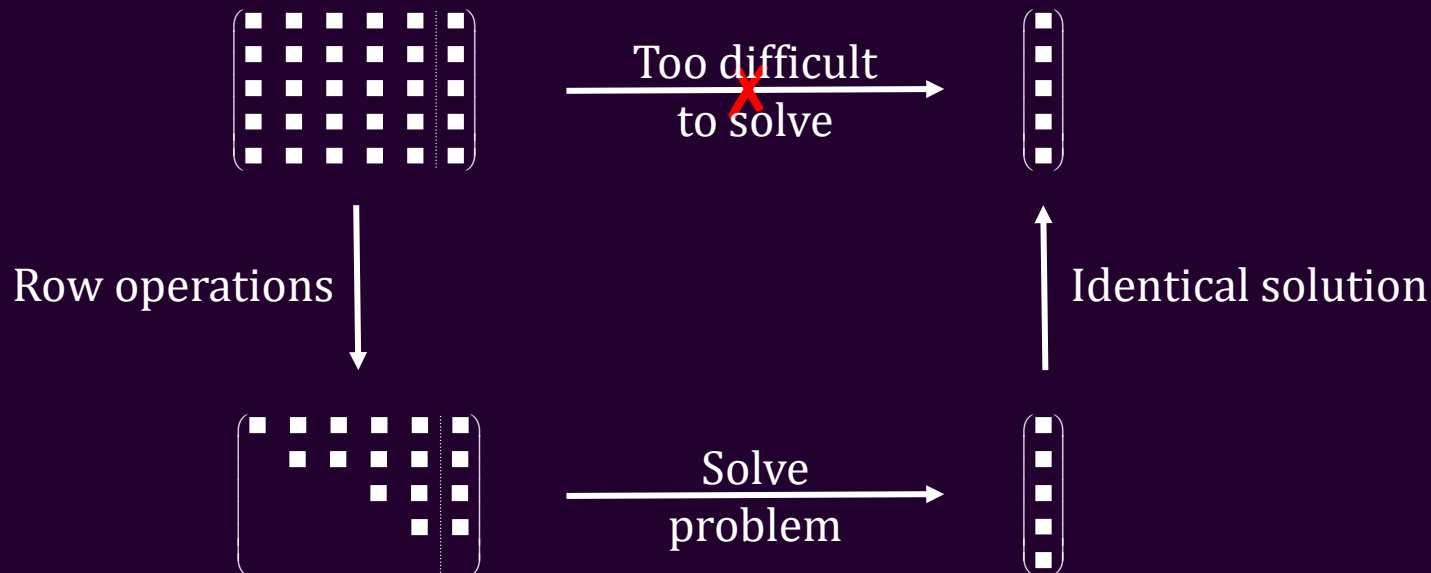
- A common problem-solving technique in engineering
 - To transform a problem that cannot be solved into an equivalent problem that can be solved
 - To solve that equivalent problem
 - To interpret that solution to find a solution to the original problem





Transforming problems

- In this case
 - Transform a matrix not in row-echelon form to one that is
 - Solve the problem in row echelon form (if a solution exists)
 - The solution is identical to that of the original problem





Swap two rows

- Recall that given a system of linear equations, there are certain acceptable operations

$$\begin{aligned} 3x_1 + 5x_2 &= 16 \\ -8x_1 + 4x_2 &= -7 \end{aligned}$$

$$\left(\begin{array}{cc|c} 3 & 5 & 16 \\ -8 & 4 & -7 \end{array} \right)$$

Swap two equations

$$\begin{aligned} -8x_1 + 4x_2 &= -7 \\ 3x_1 + 5x_2 &= 16 \end{aligned}$$

$R_{R1 \leftrightarrow R2}$

Swap two rows

$$\left(\begin{array}{cc|c} -8 & 4 & -7 \\ 3 & 5 & 16 \end{array} \right)$$





Swap two rows

- The row operation of swapping Row i and Row j is denoted by

$$R_{Ri \leftrightarrow Rj}$$

- If $(A \mid \mathbf{b})$ is an augmented matrix, and you want to show this operation applied to this augmented matrix, you can write

$$R_{Ri \leftrightarrow Rj} (A \mid \mathbf{b})$$

$$R_{R1 \leftrightarrow R3} \left(\begin{array}{ccc|c} 1.2 & 0.7 & -0.6 & 3.4 \\ 0.2 & 2.5 & 0.9 & 4.8 \\ -0.1 & -0.3 & 1.8 & -4.5 \end{array} \right) = \left(\begin{array}{ccc|c} -0.1 & -0.3 & 1.8 & -4.5 \\ 0.2 & 2.5 & 0.9 & 4.8 \\ 1.2 & 0.7 & -0.6 & 3.4 \end{array} \right)$$





Multiplying by a non-zero scalar

- Recall that given a system of linear equations, there are certain acceptable operations

$$\begin{array}{rcl} 3x_1 + 5x_2 & = & 16 \\ -8x_1 + 4x_2 & = & -7 \end{array} \quad \left(\begin{array}{cc|c} 3 & 5 & 16 \\ -8 & 4 & -7 \end{array} \right)$$

Multiplying an equation by a non-zero scalar

$R_{-3 \cdot R_2}$

$$\begin{array}{rcl} 3x_1 + 5x_2 & = & 16 \\ 24x_1 - 12x_2 & = & 21 \end{array} \quad \left(\begin{array}{cc|c} 3 & 5 & 16 \\ 24 & -12 & 21 \end{array} \right)$$

Multiplying a row by a non-zero scalar





Multiplying by a non-zero scalar

- The row operation of multiplying Row i by a non-zero scalar α is

$$R_{\alpha \cdot Ri}$$

- If $(A \mid \mathbf{b})$ is an augmented matrix, and you want to show this operation applied to this augmented matrix, you can write

$$R_{2.5R2} \left(\begin{array}{ccc|c} 1.2 & 0.7 & -0.6 & 3.4 \\ 0.2 & 2.6 & 0.4 & 4.8 \\ -0.1 & -0.3 & 1.8 & -4.5 \end{array} \right) = \left(\begin{array}{ccc|c} 1.2 & 0.7 & -0.6 & 3.4 \\ 0.5 & 6.5 & 1.0 & 12.0 \\ -0.1 & -0.3 & 1.8 & -4.5 \end{array} \right)$$





Adding a multiple of one row onto another

- Recall that given a system of linear equations, there are certain acceptable operations

$$\begin{array}{l} 3x_1 + 5x_2 = 16 \\ -8x_1 + 4x_2 = -7 \end{array} \quad \left(\begin{array}{cc|c} 3 & 5 & 16 \\ -8 & 4 & -7 \end{array} \right)$$

Adding a multiple of one equation onto another

$R_{3 \cdot R1 \rightarrow R2}$

$$\begin{array}{l} 3x_1 + 5x_2 = 16 \\ x_1 + 19x_2 = 41 \end{array} \quad \left(\begin{array}{cc|c} 3 & 5 & 16 \\ 1 & 19 & 41 \end{array} \right)$$

Adding a multiple of one row onto another





Adding a multiple of one row onto another

- The row operation of adding a multiple α of Row i onto Row j is

$$R_{\alpha \cdot Ri \rightarrow Rj}$$

- If $(A \mid \mathbf{b})$ is an augmented matrix, and you want to show this operation applied to this augmented matrix, you can write

$$R_{\alpha \cdot Ri \rightarrow Rj} (A \mid \mathbf{b})$$

$$R_{2 \cdot R3 \rightarrow R2} \left(\begin{array}{ccc|c} 1.2 & 0.7 & -0.6 & 3.4 \\ 0.2 & 2.6 & 0.4 & 4.8 \\ -0.1 & -0.3 & 1.8 & -4.5 \end{array} \right) = \left(\begin{array}{ccc|c} 1.2 & 0.7 & -0.6 & 3.4 \\ 0 & 2 & 4 & 4.2 \\ -0.1 & -0.3 & 1.8 & -4.5 \end{array} \right)$$





Row operations

- Consequently, given an augmented matrix, we have three row operations

Swapping Row i and Row j

$$R_{Ri \leftrightarrow Rj}$$

Multiply Row i by a non-zero scalar α

$$R_{\alpha \cdot Ri}$$

Adding a multiple α of Row i onto Row j

$$R_{\alpha \cdot Ri \rightarrow Rj}$$





Undoing row operations

- Can we undo a row operation?
 - That is, if we applied a row operation to an augmented matrix, is there a row operation that gets us back to the original augmented matrix?





Undoing a swap

- Suppose we have an augmented matrix $(A \mid \mathbf{b})$ and we have applied the operation of swapping Rows 3 and 5
 - To undo this, all we need do is swap Rows 3 and 5 again

$$R_{R3 \leftrightarrow R5} R_{R3 \leftrightarrow R5} (A \mid \mathbf{b}) = (A \mid \mathbf{b})$$

- This is like writing $\ln(e^x) = x$
- Thus, we may write that

$$R_{Ri \leftrightarrow Rj}^{-1} = R_{Ri \leftrightarrow Rj}$$





Undoing a scalar multiple of a row

- Suppose we have an augmented matrix $(A \parallel \mathbf{b})$ and we have applied the operation of multiplying Row 7 by -1.25
 - To undo this, all we need do is multiply Rows 7 by $(-1.25)^{-1} = -0.8$

$$R_{-0.8R7} R_{-1.25R7} (A \parallel \mathbf{b}) = (A \parallel \mathbf{b})$$

- Thus, we may write that

$$R_{\alpha \cdot Ri}^{-1} = R_{\alpha^{-1} \cdot Ri}$$





Undoing adding a scalar multiple of a row

- Suppose we have an augmented matrix $(A \mid \mathbf{b})$ and we have applied the operation of adding 2.7 times Row 1 onto Row 5
 - To undo this, all we need do is add -2.7 times Rows 1 onto Row 5

$$R_{-2.7R1 \rightarrow R5} R_{2.7R1 \rightarrow R5} (A \mid \mathbf{b}) = (A \mid \mathbf{b})$$

- Thus, we may write that

$$R_{\alpha \cdot Ri \rightarrow Rj}^{-1} = R_{-\alpha \cdot Ri \rightarrow Rj}$$





Row operations

- Thus, we have three row operations,
and the inverse of each row operation is another row operation

Swapping Row i and Row j

$$R_{Ri \leftrightarrow Rj}$$

$$R_{Ri \leftrightarrow Rj}$$

Multiply Row i by a non-zero scalar α

$$R_{\alpha \cdot Ri}$$

$$R_{\alpha^{-1} \cdot Ri}$$

Adding a multiple α of Row i onto Row j

$$R_{\alpha \cdot Ri \rightarrow Rj}$$

$$R_{-\alpha \cdot Ri \rightarrow Rj}$$





Row equivalence

- What is important about row operations is that they do not change the solution to the augmented matrix

$$R_{-0.6 \cdot R1 \rightarrow R2} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 1.8 & 3.2 & 2.4 & 6.4 \\ -2.1 & 0.2 & 8.1 & 2.4 \end{array} \right) = \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & 2 & 3 & 4 \\ -2.1 & 0.2 & 8.1 & 2.4 \end{array} \right)$$

$$R_{0.6 \cdot R1 \rightarrow R3} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & 2 & 3 & 4 \\ -2.1 & 0.2 & 8.1 & 2.4 \end{array} \right) = \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 1.6 & 7.4 & 5.2 \end{array} \right)$$

$$R_{-0.8 \cdot R2 \rightarrow R3} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 1.6 & 7.4 & 5.2 \end{array} \right) = \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 5 & 2 \end{array} \right)$$





Row equivalence

Definition:

Two augmented matrices are *row equivalent* if there exists a sequence of row operations that transforms one augmented matrix into the other.

- If two augmented matrices $(A \parallel \mathbf{b})$ and $(C \parallel \mathbf{d})$ are row equivalent, then we will write

$$(A \parallel \mathbf{b}) \sim (C \parallel \mathbf{d})$$





Row equivalence

- Recall some of the properties of equality:
 - $x = x$
 - $x = y$ if and only if $y = x$
 - if $x = y$ and $y = z$, then $x = z$
- Note that the same is for row equivalence:
 - $(A \parallel \mathbf{b}) \sim (A \parallel \mathbf{b})$
 - $(A \parallel \mathbf{b}) \sim (C \parallel \mathbf{d})$ if and only if $(C \parallel \mathbf{d}) \sim (A \parallel \mathbf{b})$
 - if $(A \parallel \mathbf{b}) \sim (C \parallel \mathbf{d})$ and $(C \parallel \mathbf{d}) \sim (E \parallel \mathbf{f})$,
then $(A \parallel \mathbf{b}) \sim (E \parallel \mathbf{f})$





Row equivalence

- When we are performing a sequence of row operations on an augmented matrix, we use this concept of row equivalence, writing

$$\begin{pmatrix} 3 & 5 & \vdots & 16 \\ -8 & 4 & \vdots & -7 \end{pmatrix} \underset{R_{R1 \leftrightarrow R2}}{\sim} \begin{pmatrix} -8 & 4 & \vdots & -7 \\ 3 & 5 & \vdots & 16 \end{pmatrix} \underset{R_{1.5 \cdot R2}}{\sim} \begin{pmatrix} -8 & 4 & \vdots & -7 \\ 4.5 & 7.5 & \vdots & 24 \end{pmatrix}$$

$$R_{1.5 \cdot R2} R_{R1 \leftrightarrow R2} \begin{pmatrix} 3 & 5 & \vdots & 16 \\ -8 & 4 & \vdots & -7 \end{pmatrix} = \begin{pmatrix} -8 & 4 & \vdots & -7 \\ 4.5 & 7.5 & \vdots & 24 \end{pmatrix}$$





Order of row operations matters

- Notice that the order in which row operations occur may affect the result
 - The results, however, are still row equivalent

$$\left(\begin{array}{cc|c} 3 & 5 & 16 \\ -8 & 4 & -7 \end{array}\right) \underset{R_{R1 \leftrightarrow R2}}{\sim} \left(\begin{array}{cc|c} -8 & 4 & -7 \\ 3 & 5 & 16 \end{array}\right) \underset{R_{-3 \cdot R2}}{\sim} \left(\begin{array}{cc|c} -8 & 4 & -7 \\ -9 & -15 & -48 \end{array}\right)$$

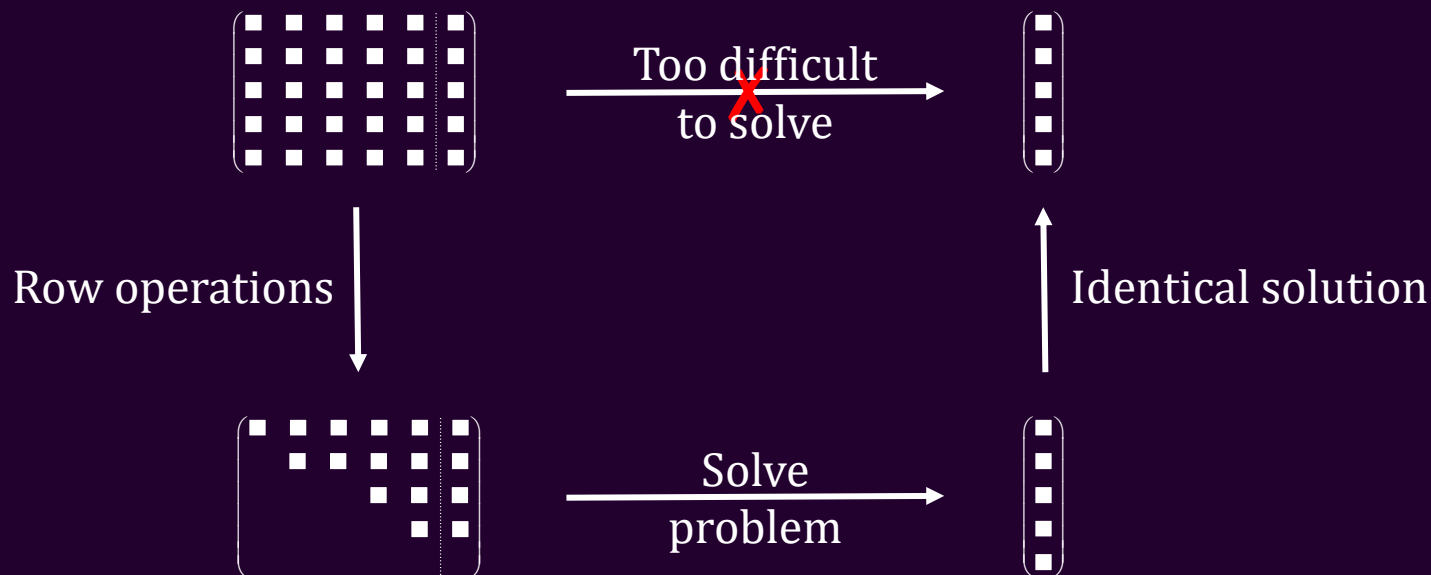
$$\left(\begin{array}{cc|c} 3 & 5 & 16 \\ -8 & 4 & -7 \end{array}\right) \underset{R_{-3 \cdot R2}}{\sim} \left(\begin{array}{cc|c} 3 & 5 & 16 \\ 24 & -12 & 21 \end{array}\right) \underset{R_{R1 \leftrightarrow R2}}{\sim} \left(\begin{array}{cc|c} 24 & -12 & 21 \\ 3 & 5 & 16 \end{array}\right)$$





Looking ahead

- Recall this diagram:
 - Given an arbitrary augmented matrix,
we will find a sequence of row operations that transforms this augmented matrix into one that is in row-echelon form
 - We will then find the solution to that augmented matrix in row-echelon form





Summary

- Following this topic, you now
 - Understand the three row operations
 - Understand that if a row operation is applied to an augmented matrix, the resulting augmented has the same solution
 - Know that each is associated with a symbol
 - Understand that each row operation has an inverse operation
 - Know the definition of row equivalence ($\sim$) and how it will be used
 - Understand that row equivalence is similar to equality in some ways





References

- [1] https://en.wikipedia.org/wiki/Elementary_matrix
- [2] https://en.wikipedia.org/wiki/Row_equivalence





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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